
Between two moments

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*Dedicated with admiration to the
memory of Jean-Marie Souriau.*

Summary. In this short note, we draw attention to a relation between two Horn polytopes which is proved in [CJ] as the result on the one side of a deep combinatorial result in [FFLP], on the other side of a simple computation involving complex structures. This suggested an inequality between Littlewood-Richardson coefficients which we prove using the symmetric characterization of these coefficients given in [CL].

1 Moment maps and Horn's problem

Recall (see for example [K]) that the mapping $H \mapsto [k \mapsto \text{Trace}(iHk)]$ is a natural identification of the space \mathcal{H} of Hermitian matrices with the dual $u(n)^*$ of the Lie algebra $u(n)$ of the unitary group $U(n)$. With this identification, the coadjoint action of $U(n)$ on $u(n)^*$ becomes the action by conjugation on the space of Hermitian matrices and the orbits \mathcal{O}_λ are uniquely characterized by the common spectrum λ of their elements. A coadjoint orbit being endowed with the Kostant-Souriau symplectic structure, the moment map becomes the canonical inclusion $\mathcal{O}_\lambda \subset \mathcal{H}$.

It follows from standard properties of moment maps (see again [K]) that with this identification, the moment map of the diagonal action of $U(n)$ on $\mathcal{O}_\lambda \times \mathcal{O}_\mu$ is simply the mapping $(A, B) \mapsto A + B$. This gives the relation with the so-called *Horn's problem* of describing the inequalities which constrain the spectrum of the sum of two hermitian matrices whose spectra are given (see the beautiful review [F]).

Kirwan's theorem asserts that, in the case of the diagonal action above, the intersection of the image of the moment map with the principal Weyl chamber $W_n^+ = \nu = \{\nu_1 \geq \nu_2 \geq \dots \geq \nu_n\}$, that is the set of ordered spectra of matrices of the form $A + B$ where $A \in \mathcal{O}_\lambda$ and $B \in \mathcal{O}_\mu$, is a convex polytope, which we shall call the *Horn polytope*. Moreover, Horn's conjecture, whose proof follows from

the works of Klyachko on the one hand, Knutson and Tao on the other hand (see [F]), gives inequalities defining the faces of the Horn polytope.

Finally, replacing Hermitian matrices by symmetric matrices and the unitary group by the orthogonal group does not change the Horn polytope, namely: the set of ordered spectra ν of the sums $A + B$ of matrices with given spectra λ and μ is the same whether A and B are hermitian or real symmetric (see [F]).

2 Complex structures and a Horn-type problem

Motivated by a study of the angular momenta of rigid body motions in dimensions higher than 3 (see [C, CJ]), let us consider the following “Horn-like” problem: characterize the set \mathcal{Q} (noted $\text{Im}\mathcal{F}$ in [CJ]) of ordered spectra of sums

$$J^{-1}S_0J + S_0,$$

where S_0 is a given $2p \times 2p$ real symmetric matrix and $J \in O(2p)$ is a complex structure, that is an isometry of \mathbb{R}^{2p} (with its standard euclidean structure) such that $J^2 = -\text{Id}$.

Replacing the set of complex structures J by the full group of isometries $O(2p)$, one gets exactly the Horn polytope in the case when $n = 2p$ and the two spectra λ and μ coincide with the one of S_0 .

The main result of [CJ] is that \mathcal{Q} is a convex polytope. The proof is not direct: one encloses \mathcal{Q} between two convex polytopes \mathcal{P}_1 and \mathcal{P}_2 associated with two Horn’s problems, one in dimension p and one in dimension $2p$, and one deduces from results due to Fomin, Fulton, Li and Poon [FFLP] and based on deep combinatorial lemmas due to Carré and Leclerc [CL], that \mathcal{P}_1 and \mathcal{P}_2 coincide, and hence coincide with \mathcal{Q} .

3 The two polytopes \mathcal{P}_1 and \mathcal{P}_2

Given decreasing sequences

$$\sigma = (\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2p}) \quad \text{and} \quad \nu = (\nu_1 \geq \cdots \geq \nu_p)$$

of real numbers, one defines

$$\begin{cases} \sigma_- = (\sigma_1, \sigma_3, \cdots, \sigma_{2p-1}), & \sigma_+ = (\sigma_2, \sigma_4, \cdots, \sigma_{2p}), \\ \nu^{(2)} = (\nu_1, \nu_1, \nu_2, \nu_2, \cdots, \nu_p, \nu_p). \end{cases} \quad (1)$$

Now, \mathcal{P}_1 is the Horn polytope for $p \times p$ symmetric matrices $a + b$ with spectrum $(a) = \sigma_-$ and spectrum $(b) = \sigma_+$, while \mathcal{P}_2 is the intersection with the set Δ of “hermitian spectra” (that is spectra of the form $\nu^{(2)}$) of the Horn polytope for $2p \times 2p$ symmetric matrices $A + B$ with spectrum $(A) = \text{spectrum}(B) = \sigma$.

4 The inclusion $\mathcal{P}_1 \subset \mathcal{P}_2$

The inclusion $\mathcal{P}_1 \subset \mathcal{P}_2$ is elementary: it follows from the inclusions

$$\mathcal{P}_1 \subset \mathcal{Q} \subset \mathcal{P}_2, \quad (2)$$

where \mathcal{Q} is the set of ordered spectra of matrices of the form $S + J^{-1}SJ$, where $S = \text{diag } \sigma$ and J is a complex structure:

$$J = R^{-1}J_0R, \quad R \in SO(2p), \quad J_0 = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}.$$

The second inclusion is obvious (replace J by any element $R \in SO(2p)$ and look for the spectra of sums $S + R^{-1}SR$ which are J -hermitian for some J ;

The first one comes from the fact that \mathcal{P}_1 is exactly the subset of $\text{Im } \mathcal{F}$ obtained when one takes into account only the complex structures J which send the subspace of \mathbb{R}^{2p} generated by the basis vectors $\mathbf{e}_1, \mathbf{e}_3, \dots, \mathbf{e}_{2p-1}$ onto the orthogonal subspace, generated by $\{\mathbf{e}_2, \mathbf{e}_4, \dots, \mathbf{e}_{2p}\}$. More precisely, it comes from the following identity where $\rho \in SO(p)$:

$$\left\{ \begin{array}{l} \begin{pmatrix} \sigma_- & 0 \\ 0 & \sigma_+ \end{pmatrix} + \begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_- & 0 \\ 0 & \sigma_+ \end{pmatrix} \begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix} \\ = \begin{pmatrix} \sigma_- + \rho^{-1}\sigma_+\rho & 0 \\ 0 & \rho\sigma_-\rho^{-1} + \sigma_+ \end{pmatrix} \end{array} \right. \quad (3)$$

5 Littlewood-Richardson coefficients

From now on, the decreasing sequences that we consider consist in non negative integers, which allows them to take the name of “partitions”. One calls $|\lambda| = \sum \lambda_i$ the “length” of the partition.

The so-called Littlewood-Richardson coefficients have their origin in the decomposition into irreducible holomorphic representations of the tensor product of two irreducible representations of the linear group $GL(n, \mathbb{C})$. For their definition and their many avatars, as well as for the computational rule expressing them in terms of the Ferrer-Young diagrams representing partitions, we refer to [F].

Let

$$\begin{aligned} \alpha &= \{\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n\}, \\ \beta &= \{\beta_1 \geq \beta_2 \geq \dots \geq \beta_n\}, \\ \gamma &= \{\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n\}, \end{aligned}$$

be three partitions.

It results from the proof of Horn’s conjecture that α, β, γ are respectively the spectra of $n \times n$ hermitian (or real symmetric) matrices A, B, C with $C = A + B$ if and only if ([F], theorem 11)

- 1) $|\gamma| = |\alpha| + |\beta|$,
- 2) the Littlewood-Richardson coefficient $c_{\alpha\beta}^\gamma$ does not vanish.

Hence, identifying Δ with a subset of \mathbb{R}^p by the mapping $\nu^{(2)} \mapsto \nu$:

$$\begin{cases} \mathcal{P}_1 = \{ \nu = (\nu_1 \geq \nu_2 \geq \dots \geq \nu_p), c_{\sigma_- \sigma_+}^\nu \neq 0 \}, \\ \mathcal{P}_2 = \{ \nu = (\nu_1 \geq \nu_2 \geq \dots \geq \nu_p), c_{\sigma \sigma}^{\nu^{(2)}} \neq 0 \}. \end{cases} \quad (4)$$

6 Yamanouchi tableaux and the Carré-Leclerc formula

For the convenience of the reader, we reproduce here definitions given in sections 4.1 and 4.2 of [FFLP].

It is classical to define a bijection Λ between strictly increasing sequences $I = (i_1 < i_2 < \dots < i_r)$ and partitions $\Lambda(I) = (\lambda_1 \geq \dots \geq \lambda_r)$ by the formulæ

$$\lambda_1 = i_r - r, \dots, \lambda_r = i_1 - 1.$$

Given two strictly increasing sequences, one defines

$$\tau(I, J) = (2i_1 - 1, \dots, 2i_r - 1) \cup (2j_1, \dots, 2j_r) \text{ reordered.}$$

To a partition λ is associated a *Young diagram of shape λ* : this is a collection of boxes arranged in rows whose lengths from top to bottom (anglo-saxon convention) are $\lambda_1, \lambda_2, \dots$. The induced operation on partitions, $(\lambda, \mu) \mapsto \tau(\lambda, \mu)$ is described in [FFLP] in terms of Young diagrams in the following way: “If one traces the Young diagram of a partition by a sequence of horizontal and vertical steps moving from Southwest to Northeast in a rectangle containing the diagrams of λ and μ , the diagram of $\tau(\lambda, \mu)$ is traced in a rectangle twice as wide in both directions, by alternating steps from λ and μ , starting with the first step of λ ”. It is also recalled in this paper that partitions $\tau(\lambda, \mu)$ are exactly the ones which correspond to *domino-decomposable* Young tableaux, that is tableaux which can be partitioned into disjoint 1×2 or 2×1 rectangles (the so-called *dominoes*). A (semi-standard) *domino tableau* is such a decomposition of a tableau into dominos with a labelling rule of the dominos similar to the one of ordinary semi-standard tableaux: labels weakly increase from left to right along lines and strictly increase along columns from top to bottom. Special domino tableaux, called *Yamanouchi*, play the leading part: let the *reading word* of a tableau be obtained by listing the labels column by column from right to left and from top to bottom (a horizontal domino is skipped the first time it is encountered); a tableau is called *Yamanouchi* if every entry i appears in any initial segment of its reading word at least as many times as any entry $j > i$. In [CL], the following rule for computing the Littlewood-Richardson coefficients is given in which, in contrast with the original Littlewood-Richardson rule, the lower indices play symmetric roles: let the *weight* ν of a Yamanouchi tableau be the sequence (ν_1, \dots, ν_n) where ν_i is the number of labels equal to i . Note that $\nu_1 \geq \dots \geq \nu_n$.

Proposition 1 ([CL] Corollary 4.4). *The coefficient $c_{\lambda\mu}^\nu$ is equal to the number of Yamanouchi domino tableaux of shape $\tau(\lambda, \mu)$ and weight ν .*

Pay attention that, while $c_{\lambda\mu}^\nu = c_{\mu\lambda}^\nu$, the diagrams $\tau(\lambda, \mu)$ and $\tau(\mu, \lambda)$ have in general different shapes. This will be important in the sequel.

7 Strengthening the inclusion $\mathcal{P}_1 \subset \mathcal{P}_2$

The following proposition strengthens the inclusion $\mathcal{P}_1 \subset \mathcal{P}_2$ while it was suggested by it:

Proposition 2. *Given partitions $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_{2p}\}$ and $\nu = \{\nu_1, \nu_2, \dots, \nu_p\}$, we have*

$$c_{\sigma_-\sigma_+}^\nu \leq c_{\sigma\sigma}^{\nu^{(2)}}.$$

Proof. As in [FFLP], we use Proposition 1. We must define an injection of the set of Yamanouchi domino tableaux of shape $\tau(\sigma_-, \sigma_+)$ (or of shape $\tau(\sigma_+, \sigma_-)$) and weight ν into the set of Yamanouchi domino tableaux of shape $\tau(\sigma, \sigma)$ and weight $\nu^{(2)}$. In order to prove the above lemma, the choice of $\tau(\sigma_+, \sigma_-)$ is much more convenient because it is simply related to $\tau(\sigma, \sigma)$ while this is not the case of $\tau(\sigma_-, \sigma_+)$, more precisely, if $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{2p})$,

$$\begin{aligned}\tau(\sigma_+, \sigma_-) &= (2\sigma_1, 2\sigma_2, \dots, 2\sigma_{2p}), \\ \tau(\sigma, \sigma) &= (2\sigma_1, 2\sigma_1, 2\sigma_2, 2\sigma_2, \dots, 2\sigma_{2p}, 2\sigma_{2p}).\end{aligned}$$

In words, the tableau $\tau(\sigma, \sigma)$ is obtained from the tableau $\tau(\sigma_+, \sigma_-)$ by duplicating each line. The injection we are looking for is then simply obtained by dividing vertically each domino of $\tau(\sigma_+, \sigma_-)$ and numbering the resulting two pieces respectively $2k-1$ and $2k$ if the original domino was numbered k . One checks immediately that the Yamanouchi property is verified. An example is given in §10 below, see Figure 1 and Figure 2.

8 Concluding remarks

1) As was said at the beginning, the inclusion $\mathcal{P}_1 \supset \mathcal{P}_2$, and hence the equality $\mathcal{P}_1 = \mathcal{P}_2$, is proved in [CJ]. It results from the stronger inclusion in \mathcal{P}_1 of the orthogonal projection on Δ of the Horn polytope

$$\mathcal{P} = \{\gamma = (\gamma_1 \geq \dots \geq \gamma_{2p}), c_{\sigma\sigma}^\gamma \neq 0\}.$$

This last inclusion follows from the inequalities $c_{\lambda\mu}^\nu \leq c_{\tau(\lambda,\mu)\tau(\lambda,\mu)}^{\tau(\nu,\nu)}$, valid as soon as the lengths of the partitions at stake satisfy $|\nu| = |\lambda| + |\mu|$, ([FFLP] Proposition 4.5). The main ingredient of the proof is again the symmetric characterization of the Littlewood-Richardson coefficients in terms of Yamanouchi tableaux given in [CL].

The equality $\mathcal{P}_1 = \mathcal{P}_2$ of the two polytopes has the consequence, not implied by the inequality of lemma 2, that

$$c_{\sigma\sigma}^{\nu^{(2)}} \neq 0 \quad \text{implies} \quad c_{\sigma_-\sigma_+}^\nu \neq 0.$$

Nevertheless, the inequality given by Proposition 2 may be strict. An example is shown in §10, Figure 3.

2) Dividing σ into two partitions λ and μ of length p in an arbitrary way leads to polytopes strictly smaller than $\mathcal{P}_1 = \mathcal{P}_2$. Correspondingly, we have the following inequalities strengthening these inclusions:

$$c_{\lambda\mu}^\nu \leq c_{\sigma_-\sigma_+}^\nu.$$

These inequalities are a special case of Corollary 14 of [LPP].

9 Questions

- 1) Find a more conceptual relation between the complex structures and the doubling of Young tableaux.
- 2) Find a direct proof of the fact that \mathcal{Q} is convex, resp. a convex polytope.
- 3) Find a relation between the inequality which is the object of Proposition 2 and the one in [FFLP] (Proposition 4.5). In the first case, each domino is divided into 2 pieces while in the second one it is divided into 4 pieces.

10 Examples

We take $\sigma = (5, 3, 2, 0)$, so that $\sigma_+ = (3, 0)$ and $\sigma_- = (5, 2)$. We then have

$$\tau(\sigma_+, \sigma_-) = (10, 6, 4, 0), \quad \tau(\sigma, \sigma) = (10, 10, 6, 6, 4, 4, 0, 0).$$

In Figure 1 below, we display the Yamanouchi domino tableaux T_1, T_2, T_3, T_4 of shape $\tau(\sigma_+, \sigma_-)$, and of respective weights

$$\nu_1 = (5, 5), \quad \nu_2 = (6, 4), \quad \nu_3 = (7, 3), \quad \nu_4 = (8, 2).$$

Their respective reading words are

$$w_1 = 1112212212, \quad w_2 = 1112212112, \quad w_3 = 1112112112, \quad w_4 = 1111112112.$$

In Figure 2 we display the corresponding Yamanouchi domino tableaux U_1, U_2, U_3, U_4 of shape $\tau(\sigma, \sigma)$, obtained from T_1, T_2, T_3, T_4 via the duplication procedure of Proposition 2.

Finally, Figure 3 gives an example which shows that the inequality of Proposition 2 may be strict. Here we take $\sigma = (7, 6, 4, 3)$ and $\nu = (10, 8, 2)$, and we exhibit a Yamanouchi domino tableau T of shape $\tau(\sigma, \sigma)$ and weight $\nu^{(2)}$ which cannot be obtained from a Yamanouchi domino tableau of shape $\tau(\sigma_+, \sigma_-)$ and weight ν by means of the duplication procedure described in the proof of Proposition 2.

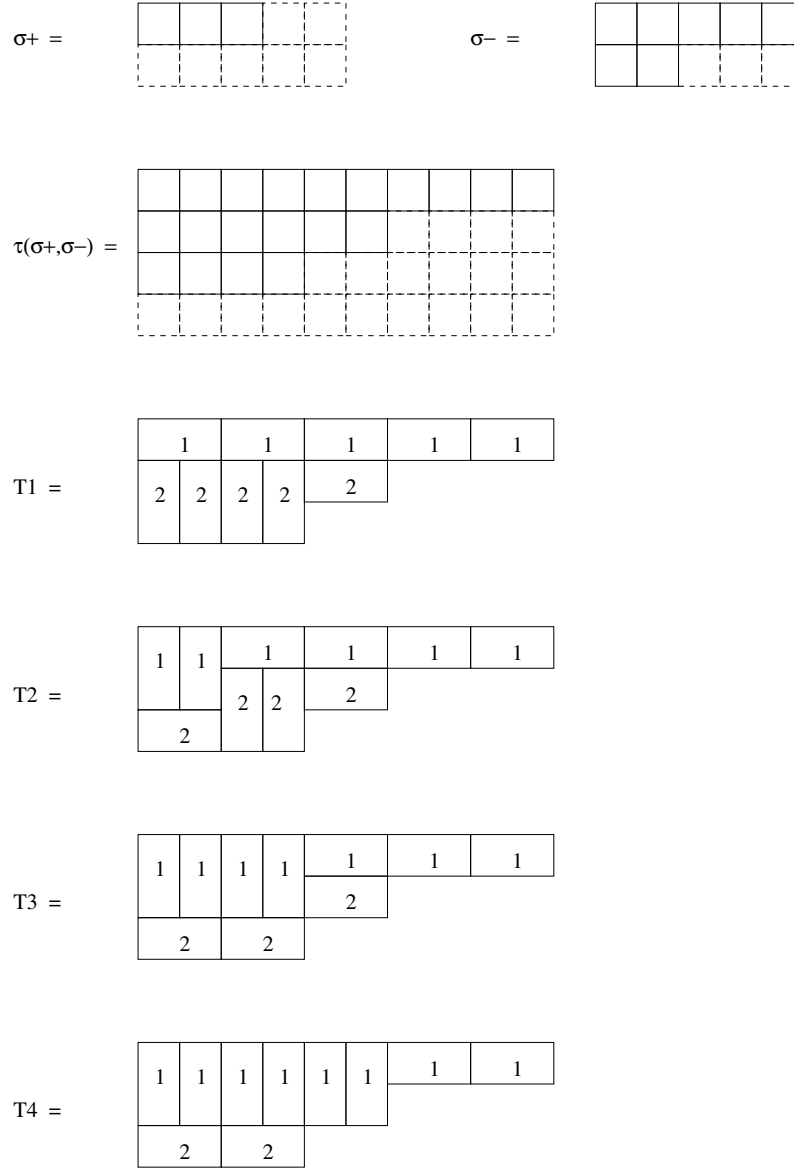


Fig. 1. Yamanouchi domino tableaux of shape $\tau(\sigma_+, \sigma_-)$.

U1 =

1		1		1		1		1
2		2		2		2		2
3	3	3	3	3	3			
					4			
4	4	4	4					

U2 =

1	1		1		1		1	
			2		2		2	
2	2	3	3	3	3			
					4			
	3		4	4				
	4							

U3 =

1	1	1	1	1	1		1	
					2		2	
2	2	2	2	2	3			
					4			
	3		3					
	4		4					

U4 =

1	1	1	1	1	1	1	1	
							2	
2	2	2	2	2	2	2		
	3		3					
	4		4					

Fig. 2. Yamanouchi domino tableaux of shape $\tau(\sigma, \sigma)$.

1	1	1	1	1	1	1	1	1	1	1
							2	2	2	2
2	2	2	2	2	2	3	3	3	3	
								4	4	
3	3	3	3		4		4			
					5		5			
4	4	4	4	6	6					

Fig. 3. A Yamanouchi tableau of shape $\tau(\sigma, \sigma)$ and weight $\nu^{(2)}$ which is not obtained by the duplication procedure of Proposition 2.

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